Computational Ocean Acoustics
(Wave Propagation Theory)

SALMAN IJAZ SIDDIQUI

Department of Electronics and Telecommunication,
Norwegian University of Science and Technology (NTNU).
NO-7491 Trondheim, Norway.

NTNU

March 26, 2015
Presentation Outline

- Wave Equation.
- Homogeneous Media.
- Layered Media and waveguides.
- Reflection and transmission co-efficients.
- Ideal Fluid waveguide.
- Deep Ocean waveguide and WKB approximation.
Wave Equation.

Homogeneous Media.

Layered Media and waveguides.

Reflection and transmission co-efficients.

Ideal Fluid waveguide.

Deep Ocean waveguide and WKB approximation.
Presentation Outline

- Wave Equation.
- Homogeneous Media.
  - Layered Media and waveguides.
  - Reflection and transmission co-efficients.
  - Ideal Fluid waveguide.
  - Deep Ocean waveguide and WKB approximation.
Presentation Outline

- Wave Equation.
- Homogeneous Media.
- Layered Media and waveguides.
  - Reflection and transmission co-efficients.
  - Ideal Fluid waveguide.
  - Deep Ocean waveguide and WKB approximation.
Presentation Outline

- Wave Equation.
- Homogeneous Media.
- Layered Media and waveguides.
- Reflection and transmission co-efficients.
- Ideal Fluid waveguide.
- Deep Ocean waveguide and WKB approximation.
Presentation Outline

- Wave Equation.
- Homogeneous Media.
- Layered Media and waveguides.
- Reflection and transmission co-efficients.
- Ideal Fluid waveguide.
- Deep Ocean waveguide and WKB approximation.
Presentation Outline

- Wave Equation.
- Homogeneous Media.
- Layered Media and waveguides.
- Reflection and transmission co-efficients.
- Ideal Fluid waveguide.
- Deep Ocean waveguide and WKB approximation.
Propagation of the acoustic wave in the fluid is governed by the wave equation.

Wave equation can be formulated by three equations:
- Equation of continuity
- Euler’s equation
- Equation of state
Propagation of the acoustic wave in the fluid is governed by the wave equation.

Wave equation can be formulated by three equations:
- Equation of continuity
- Euler’s equation
- Equation of state
Propagation of the acoustic wave in the fluid is governed by the wave equation.

Wave equation can be formulated by three equations:

- Equation of continuity
- Euler’s equation
- Equation of state
Wave Equation

- Propagation of the acoustic wave in the fluid is governed by the wave equation.

- Wave equation can be formulated by three equations:
  - Equation of continuity
  - Euler’s equation
  - Equation of state
Propagation of the acoustic wave in the fluid is governed by the wave equation.

Wave equation can be formulated by three equations:

- Equation of continuity
- Euler’s equation
- Equation of state
Propagation of the acoustic wave in the fluid is governed by the wave equation.

Wave equation can be formulated by three equations:
- Equation of continuity
- Euler’s equation
- Equation of state
Wave equation

- **Continuity Equation**: Net change in the mass due to its flow through an element is equal to the changes in the density of the mass of element.

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v)
\]

- **Euler’s Equation**: Force equals mass time acceleration.

\[
\rho \left[ \frac{\partial v}{\partial t} + v \cdot \nabla v \right] = -\nabla p
\]

- **State Equation**: Relationship between the change in density and a change in pressure.

\[
p = p(\rho, S) \text{ where } S \text{ is the entropy}
\]
Continuity Equation: Net change in the mass due to its flow through an element is equal to the changes in the density of the mass of element.

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v) \]

Euler’s Equation: Force equals mass time acceleration.

\[ \rho \left[ \frac{\partial v}{\partial t} + v \cdot \nabla v \right] = -\nabla p \]

State Equation: Relationship between the change in density and a change in pressure.

\[ p = p(\rho, S) \text{ where } S \text{ is the entropy} \]
Wave equation

- **Continuity Equation**: Net change in the mass due to its flow through an element is equal to the changes in the density of the mass of element.

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v)
\]

- **Euler’s Equation**: Force equals mass time acceleration.

\[
\rho \left[ \frac{\partial v}{\partial t} + v \cdot \nabla v \right] = -\nabla p
\]

- **State Equation**: Relationship between the change in density and a change in pressure.

\[
p = p(\rho, S) \text{ where } S \text{ is the entropy}
\]
- **Continuity Equation**: Net change in the mass due to its flow through an element is equal to the changes in the density of the mass of element.

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})
\]

- **Euler’s Equation**: Force equals mass time acceleration.

\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = -\nabla p
\]

- **State Equation**: Relationship between the change in density and a change in pressure.

\[
p = p(\rho, S) \text{ where } S \text{ is the entropy}
\]
Linear Wave equation

- Assuming that each physical quantity is a function of a steady-state, time-independent value and a small fluctuation term

\[
\begin{align*}
  p &= p_0 + p' \\
  \rho &= \rho_0 + \rho' \\
  v &= v_0 + v'
\end{align*}
\]

- The linearized equations are

\[
\begin{align*}
  \frac{\partial \rho'}{\partial t} &= - \nabla \cdot (\rho_0 v) \\
  \frac{\partial v}{\partial t} &= - \frac{1}{\rho_0} \nabla p' (\rho) \\
  \frac{\partial p'}{\partial t} &= c^2 \left( \frac{\partial \rho'}{\partial t} + v \cdot \nabla \rho_0 \right)
\end{align*}
\]
Assuming that each physical quantity is a function of a steady-state, time-independent value and a small fluctuation term

\[ p = p_0 + p' \]
\[ \rho = \rho_0 + \rho' \]
\[ v = v_0 + v' \]

The linearized equations are

\[ \frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 v) \]
\[ \frac{\partial v}{\partial t} = -\frac{1}{\rho_0} \nabla p'(\rho) \]
\[ \frac{\partial p'}{\partial t} = c^2 \left( \frac{\partial \rho'}{\partial t} + v \cdot \nabla \rho_0 \right) \]
Linear Wave equation

- Assuming that each physical quantity is a function of a steady-state, time-independent value and a small fluctuation term

\[ p = p_0 + p' \]
\[ \rho = \rho_0 + \rho' \]
\[ v = v_0 + v' \]

- The linearized equations are

\[ \frac{\partial p'}{\partial t} = -\nabla \cdot (\rho_0 v) \]
\[ \frac{\partial v}{\partial t} = -\frac{1}{\rho_0} \nabla p'(\rho) \]
\[ \frac{\partial p'}{\partial t} = c^2 \left( \frac{\partial p'}{\partial t} + v \cdot \nabla \rho_0 \right) \]
Different forms of Wave equation

- Wave equation for pressure

\[ \rho \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \]

- Wave equation for particle velocity

\[ \frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot v) - \frac{\partial^2 v}{\partial t^2} = 0 \]

- Wave equation for velocity potential \((v = \nabla \phi)\)

\[ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \]
Different forms of Wave equation

- Wave equation for pressure

\[ \rho \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \]

- Wave equation for particle velocity

\[ \frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot v) - \frac{\partial^2 v}{\partial t^2} = 0 \]

- Wave equation for velocity potential \((v = \nabla \phi)\)

\[ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \]
Different forms of Wave equation

- Wave equation for pressure

\[ \rho \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \]

- Wave equation for particle velocity

\[ \frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot v) - \frac{\partial^2 v}{\partial t^2} = 0 \]

- Wave equation for velocity potential \((v = \nabla \phi)\)

\[ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \]
Different forms of Wave equation

- Wave equation for pressure

\[ \rho \nabla \cdot \left( \frac{1}{\rho} \nabla p \right) - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \]

- Wave equation for particle velocity

\[ \frac{1}{\rho} \nabla (\rho c^2 \nabla \cdot v) - \frac{\partial^2 v}{\partial t^2} = 0 \]

- Wave equation for velocity potential (\( v = \nabla \phi \))

\[ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \]
Different forms of Wave equation

- Wave equation for Displacement potential ($u = \nabla \psi$)

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$$

- Wave equation in the presence of a source

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = f(r, t)$$
Different forms of Wave equation

- Wave equation for Displacement potential \((u = \nabla \psi)\)

\[ \nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \]

- Wave equation in the presence of a source

\[ \nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = f(r, t) \]
Different forms of Wave equation

- Wave equation for Displacement potential \( u = \nabla \psi \)

\[
\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0
\]

- Wave equation in the presence of a source

\[
\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = f(r, t)
\]
Helmholtz equation

- Wave equation in the frequency domain.
- Obtained by taking the Fourier transform of the time domain wave equation.

\[
[\nabla^2 + k^2(r)]\psi(r, \omega) = f(r, \omega) \quad \text{where} \quad k(r) = \frac{\omega}{c(r)}
\]

- Not suitable to broadband applications due to the complexity of obtaining the inverse fourier transform.
Wave equation in the frequency domain.

Obtained by taking the Fourier transform of the time domain wave equation.

\[
[\nabla^2 + k^2(r)]\psi(r, \omega) = f(r, \omega) \quad \text{where} \quad k(r) = \frac{\omega}{c(r)}
\]

Not suitable to broadband applications due to the complexity of obtaining the inverse Fourier transform.
Helmholtz equation

- Wave equation in the frequency domain.
- Obtained by taking the Fourier transform of the time domain wave equation.

\[
[\nabla^2 + k^2(r)]\psi(r, \omega) = f(r, \omega) \quad \text{where} \quad k(r) = \frac{\omega}{c(r)}
\]

- Not suitable to broadband applications due to the complexity of obtaining the inverse fourier transform.
Helmholtz equation

- Wave equation in the frequency domain.
- Obtained by taking the Fourier transform of the time domain wave equation.

\[
[\nabla^2 + k^2(r)]\psi(r, \omega) = f(r, \omega) \quad \text{where} \quad k(r) = \frac{\omega}{c(r)}
\]

- Not suitable to broadband applications due to the complexity of obtaining the inverse fourier transform.
Solution of the wave equation

- Three dimensional, elliptical partial differential equation.
- No universal solution available.
- The solution depends on the following factors:
  - Dimensionality of the problem
  - Sound speed variations $c(r)$
  - Boundary conditions
  - Source-receiver geometry
  - Frequency and bandwidth
- Optimum approach is the hybridization of analytical and numerical methods.
Solution of the wave equation

- Three dimensional, elliptical partial differential equation.

- No universal solution available.

- The solution depends on the following factors:
  - Dimensionality of the problem
  - Sound speed variations $c(r)$
  - Boundary conditions
  - Source-receiver geometry
  - Frequency and bandwidth

- Optimum approach is the hybridization of analytical and numerical methods.
Solution of the wave equation

- Three dimensional, elliptical partial differential equation.
- No universal solution available.
- The solution depends on the following factors:
  - Dimensionality of the problem
  - Sound speed variations $c(r)$
  - Boundary conditions
  - Source-receiver geometry
  - Frequency and bandwidth
- Optimum approach is the hybridization of analytical and numerical methods.
Solution of the wave equation

- Three dimensional, elliptical partial differential equation.
- No universal solution available.
- The solution depends on the following factors:
  - Dimensionality of the problem
  - Sound speed variations $c(r)$
  - Boundary conditions
  - Source-receiver geometry
  - Frequency and bandwidth
- Optimum approach is the hybridization of analytical and numerical methods.
Solution of the wave equation

- Three dimensional, elliptical partial differential equation.
- No universal solution available.
- The solution depends on the following factors:
  - Dimensionality of the problem
  - Sound speed variations $c(r)$
  - Boundary conditions
  - Source-receiver geometry
  - Frequency and bandwidth
- Optimum approach is the hybridization of analytical and numerical methods.
Wave equation solution in different co-ordinate systems

- Cartesian coordinates

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ \psi(x, y, z) = \begin{cases} 
A e^{ik \cdot r} \\
B e^{-ik \cdot r} 
\end{cases} \]

where \( k = (k_x, k_y, k_z) \) is the wave vector and A, B are arbitrary amplitudes
Wave equation solution in different co-ordinate systems

- Cartesian coordinates

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ \psi(x, y, z) = \begin{cases} 
    Ae^{ik \cdot r} \\
    Be^{-ik \cdot r} 
\end{cases} \]

where \( k = (k_x, k_y, k_z) \) is the wave vector and A, B are arbitrary amplitudes
Wave equation solution in different co-ordinate systems

- Cylindrical coordinates

\[
\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}
\]

- For uniform line source, the solution of the wave equations is

\[
\psi(r) = \begin{cases} 
CH_0^{(1)}(kr) \\
DH_0^{(2)}(kr)
\end{cases}
\]

where \(H_0^{(1)}, H_0^{(2)}\) are the hankel functions which can be represented in the asymptotic form as

\[
H_0^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi kr}} e^{i(kr-\pi/4)}
\]

\[
H_0^{(2)}(kr) \simeq \sqrt{\frac{2}{\pi kr}} e^{-i(kr-\pi/4)}
\]
Wave equation solution in different co-ordinate systems

- Cylindrical coordinates

\[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \]

- For uniform line source, the solution of the wave equations is

\[ \psi(r) = \begin{cases} 
CH_0^{(1)}(kr) \\
DH_0^{(2)}(kr) 
\end{cases} \]

where \( H_0^{(1)}, H_0^{(2)} \) are the hankel functions which can be represented in the asymptotic form as

\[ H_0^{(1)}(kr) \approx \sqrt{\frac{2}{\pi kr}} e^{i(kr - \pi/4)} \]

\[ H_0^{(2)}(kr) \approx \sqrt{\frac{2}{\pi kr}} e^{-i(kr - \pi/4)} \]
Wave equation solution in different co-ordinate systems

- Cylindrical coordinates

\[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \]

- For uniform line source, the solution of the wave equations is

\[ \psi(r) = \begin{cases} CH_0^{(1)}(kr) \\ DH_0^{(2)}(kr) \end{cases} \]

where \( H_0^{(1)}, H_0^{(2)} \) are the hankel functions which can be represented in the asymptotic form as

\[ H_0^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi kr}} e^{i(\pi - \pi/4)} \]
\[ H_0^{(2)}(kr) \simeq \sqrt{\frac{2}{\pi kr}} e^{-i(\pi - \pi/4)} \]
Spherical Co-ordinates

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

\[ \psi(r) = \begin{cases} 
  (A/r)e^{ik \cdot r} \\
  (B/r)e^{-ik \cdot r} 
\end{cases} \]

where \( k \) is the wave vector and \( A, B \) are arbitrary amplitudes.
Spherical Co-ordinates

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

\[ \psi(r) = \begin{cases} (A/r)e^{ik \cdot r} \\ (B/r)e^{-ik \cdot r} \end{cases} \]

where \( k \) is the wave vector and \( A, B \) are arbitrary amplitudes
Source in an unbounded medium

- Assuming an acoustic field in a homogeneous fluid due to a small sphere of radius $a$ with a surface displacement given as

$$u_r(t, a) = U(t)$$

- In the homogeneous fluid, the field will be omni-directional, with the radial displacement

$$u_r = \frac{\partial \psi(r, t)}{\partial r}$$
Source in an unbounded medium

Assuming an acoustic field in a homogeneous fluid due to a small sphere of radius $a$ with a surface displacement given as

$$u_r(t, a) = U(t)$$

In the homogeneous fluid, the field will be omni-directional, with the radial displacement

$$u_r = \frac{\partial \psi(r, t)}{\partial r}$$
Source in an unbounded medium

Assuming an acoustic field in a homogeneous fluid due to a small sphere of radius \( a \) with a surface displacement given as

\[ u_r(t, a) = U(t) \]

In the homogeneous fluid, the field will be omni-directional, with the radial displacement

\[ u_r = \frac{\partial \psi(r, t)}{\partial r} \]
where $\psi$ is the displacement potential.

Taking the fourier transform and applying the boundary condition, the solution the displacement field becomes

$$\psi(r) = -S_\omega \frac{e^{ikr}}{4\pi r}$$

where $S_\omega$ is the source strength.
where $\psi$ is the displacement potential.

Taking the fourier transform and applying the boundary condition, the solution the displacement field becomes

$$\psi(r) = -S_\omega \frac{e^{ikr}}{4\pi r}$$

where $S_\omega$ is the source strength.
where $\psi$ is the displacement potential.

Taking the fourier transform and applying the boundary condition, the solution the displacement field becomes

$$\psi(r) = -S_\omega \frac{e^{ikr}}{4\pi r}$$

where $S_\omega$ is the source strength.
Green Function

- Green function can be defined as the behavior of the channel. In frequency domain it is given by

\[ g_\omega(r, r_0) = \frac{e^{ikR}}{4\pi R} \quad \text{where} \quad R = |r - r_0| \]

- Green function satisfies the inhomogeneous Helmholtz equation,

\[ [\nabla^2 + k^2]g_\omega(r, r_0) = -\delta(r - r_0) \]

- The green function for the time domain wave equation is obtained by taking the Fourier transform

\[ g_t(r, r_0) = \frac{\delta(R/c - t)}{4\pi R} \]
Green function can be defined as the behavior of the channel. In frequency domain it is given by

\[ g_\omega(r, r_0) = \frac{e^{ikR}}{4\pi R} \quad \text{where} \quad R = |r - r_0| \]

Green function satisfies the inhomogeneous Helmholtz equation,

\[ [\nabla^2 + k^2]g_\omega(r, r_0) = -\delta(r - r_0) \]

The green function for the time domain wave equation is obtained by taking the Fourier transform

\[ g_t(r, r_0) = \frac{\delta(R/c - t)}{4\pi R} \]
Green Function

- Green function can be defined as the behavior of the channel. In frequency domain it is given by

\[ g_\omega(r, r_0) = \frac{e^{ikR}}{4\pi R} \quad \text{where} \quad R = |r - r_0| \]

- Green function satisfies the inhomogeneous Helmholtz equation,

\[ [\nabla^2 + k^2]g_\omega(r, r_0) = -\delta(r - r_0) \]

- The green function for the time domain wave equation is obtained by taking the Fourier transform

\[ g_t(r, r_0) = \frac{\delta(R/c - t)}{4\pi R} \]
Green Function

- Green function can be defined as the behavior of the channel. In frequency domain it is given by

\[ g_\omega(r, r_0) = \frac{e^{ikR}}{4\pi R} \quad \text{where} \quad R = |r - r_0| \]

- Green function satisfies the inhomogeneous Helmholtz equation,

\[ [\nabla^2 + k^2]g_\omega(r, r_0) = -\delta(r - r_0) \]

- The green function for the time domain wave equation is obtained by taking the Fourier transform

\[ g_t(r, r_0) = \frac{\delta(R/c - t)}{4\pi R} \]
In case of more realistic environment like a bounded medium, the green function satisfies the Helmholtz equation by

$$[\nabla^2 + k^2]G_\omega(r, r_0) = \delta(r - r_0)$$

$$G_\omega(r, r_0) = g_\omega(r, r_0) + H_\omega(r)$$

Using the green function for the bounded medium and doing some mathematical manipulations
In case of more realistic environment like a bounded media, the green function satisfies the Helmholtz equation by

\[ \nabla^2 + k^2 \] \[ G_\omega(r, r_0) = \delta(r - r_0) \]

\[ G_\omega(r, r_0) = g_\omega(r, r_0) + H_\omega(r) \]

Using the green function for the bounded medium and doing some mathematical manipulations
In case of more realistic environment like a bounded media, the green function satisfies the Helmholtz equation by

\[ \nabla^2 + k^2 G_\omega(r, r_0) = \delta(r - r_0) \]
\[ G_\omega(r, r_0) = g_\omega(r, r_0) + H_\omega(r) \]

Using the green function for the bounded medium and doing some mathematical manipulations.
Source in a bounded medium

\[ \psi(r) = \int_S \left[ G_\omega(r, r_0) \frac{\partial \psi(r_0)}{\partial n_0} - \psi(r_0) \frac{\partial G_\omega(r, r_0)}{\partial n_0} \right] dS_0 \]

\[ - \int_V f(r_0) G_\omega(r, r_0) dV_0 \]

where \( n_0 \) is the outward pointing normal on the surface.

- Difficult to attain the closed form solution.
- Make assumptions on the green function to simplify the expression.
Source in a bounded medium

\[
\psi(r) = \int_S \left[ G_\omega(r,r_0) \frac{\partial \psi(r_0)}{\partial n_0} - \psi(r_0) \frac{\partial G_\omega(r,r_0)}{\partial n_0} \right] dS_0
\]

\[
- \int_V f(r_0) G_\omega(r,r_0) dV_0
\]

where \( n_0 \) is the outward pointing normal on the surface.

- Difficult to attain the closed form solution.
- Make assumptions on the green function to simplify the expression.
Source in a bounded medium

\[ \psi(r) = \int_S \left[ G_\omega(r, r_0) \frac{\partial \psi(r_0)}{\partial n_0} - \psi(r_0) \frac{\partial G_\omega(r, r_0)}{\partial n_0} \right] dS_0 \]

\[ - \int_V f(r_0) G_\omega(r, r_0) dV_0 \]

where \( n_0 \) is the outward pointing normal on the surface.

- Difficult to attain the closed form solution.
- Make assumptions on the green function to simplify the expression.
Assuming a point source is placed at \( r_s = (x_s, y_s, z_s) \) and origin at the surface, we can replace the pressure release boundary condition by

\[
\psi(r_0) = 0, \quad r_0 = (x, y, 0)
\]

Using the green function for the bounded media we have

\[
\psi(r) = \int_S G_\omega(r, r_0) \frac{\partial \psi(r_0)}{\partial n_0} - \int_V f(r_0) G_\omega(r, r_0) dV_0
\]
Assuming a point source is placed at \( r_s = (x_s, y_s, z_s) \) and origin at the surface, we can replace the pressure release boundary condition by

\[
\psi(r_0) = 0, \quad r_0 = (x, y, 0)
\]

Using the green function for the bounded media we have

\[
\psi(r) = \int_S G_\omega(r, r_0) \frac{\partial \psi(r_0)}{\partial n_0} - \int_V f(r_0)G_\omega(r, r_0) dV_0
\]
Point source in fluid halfspace

Assuming a point source is placed at \( r_s = (x_s, y_s, z_s) \) and origin at the surface, we can replace the pressure release boundary condition by

\[
\psi(r_0) = 0, \quad r_0 = (x, y, 0)
\]

Using the green function for the bounded media we have

\[
\psi(r) = \int_S G_\omega(r, r_0) \frac{\partial \psi(r_0)}{\partial n_0} - \int_V f(r_0) G_\omega(r, r_0) dV_0
\]
Point source in fluid halfspace

where \( n_0 \) is the outward pointing normal on the surface.

For a point source, the source field is given by

\[
f(r_0) = S_\omega \delta(r_0 - r_s)
\]

In order to simplify, we can choose the green function such that \( G_\omega(r, r_0) = 0 \) then the displacement potential becomes

\[
\psi(r) = -S_\omega G_\omega(r, r_s)
\]
where $n_0$ is the outward pointing normal on the surface.

For a point source, the source field is given by

$$f(r_0) = S_\omega \delta(r_0 - r_s)$$

In order to simplify, we can choose the green function such that $G_\omega(r, r_0) = 0$ then the displacement potential becomes

$$\psi(r) = -S_\omega G_\omega(r, r_s)$$
where \( n_0 \) is the outward pointing normal on the surface.

For a point source, the source field is given by

\[
 f(r_0) = S_\omega \delta(r_0 - r_s)
\]

In order to simplify, we can choose the green function such that \( G_\omega(r, r_0) = 0 \) then the displacement potential becomes

\[
 \psi(r) = -S_\omega G_\omega(r, r_s)
\]
where $n_0$ is the outward pointing normal on the surface.

For a point source, the source field is given by

$$f(r_0) = S_\omega \delta(r_0 - r_s)$$

In order to simplify, we can choose the green function such that $G_\omega(r, r_0) = 0$ then the displacement potential becomes

$$\psi(r) = -S_\omega G_\omega(r, r_s)$$
Integral Transform Technique

- Applicable when both the coefficients of the Helmholtz equation and boundary conditions are independent of one or more space coordinates.
- Choice of the coordinate system is important.
- Boundary conditions control the choice of the coordinate system.
Integral Transform Technique

- Applicable when both the coefficients of the Helmholtz equation and boundary conditions are independent of one or more space coordinates.
- Choice of the coordinate system is important.
- Boundary conditions control the choice of the coordinate system.
Integral Transform Technique

- Applicable when both the coefficients of the Helmholtz equation and boundary conditions are independent of one or more space coordinates.

- Choice of the coordinate system is important.

- Boundary conditions control the choice of the coordinate system.
Integral Transform Technique

- Applicable when both the coefficients of the Helmholtz equation and boundary conditions are independent of one or more space coordinates.
- Choice of the coordinate system is important.
- Boundary conditions control the choice of the coordinate system.
Plane propagation problem

- Choosing the Cartesian coordinates and assuming an infinite line source along y-axis, the Helmholtz equation is reduced to two dimension, the range and the depth.

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(z) \right] \psi(x, z) = S_\omega \delta(x) \delta(z - z_s)
\]

- The boundary condition is given by

\[
B[\psi(r)]_{z=z_n} = 0, \quad n = 1, \ldots, N
\]

- Taking the Fourier transform to obtain the depth separated equation and inserting the value of the displacement potential,

\[
\left[ \frac{d^2}{dz^2} + (k^2 - k_x^2) \right] G_\omega(k_x, z, z_s) = -\frac{\delta(z - z_s)}{2\pi}
\]

- \(G_\omega(k_x, z, z_s)\) is the depth dependent Green function, which is the superposition of the free field and homogeneous field.

- The total field must satisfy the boundary conditions.
Plane propagation problem

- Choosing the Cartesian coordinates and assuming an infinite line source along y-axis, the Helmholtz equation is reduced to two dimension, the range and the depth.

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(z) \right] \psi(x, z) = S_\omega \delta(x) \delta(z - z_s)
\]

- The boundary condition is given by

\[ B[\psi(r)]_{z=z_n} = 0, \quad n = 1, \ldots, N \]

- Taking the Fourier transform to obtain the depth separated equation and inserting the value of the displacement potential,

\[
\left[ \frac{d^2}{dz^2} + (k^2 - k^2_x) \right] G_\omega(k_x, z, z_s) = -\frac{\delta(z - z_s)}{2\pi}
\]

- \( G_\omega(k_x, z, z_s) \) is the depth dependent Green function, which is the superposition of the free field and homogeneous field.

- The total field must satisfy the boundary conditions.
Plane propagation problem

- Choosing the Cartesian coordinates and assuming an infinite line source along y-axis, the Helmholtz equation is reduced to two dimension, the range and the depth.

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(z) \psi(x, z) = S_\omega \delta(x) \delta(z - z_s)
\]

- The boundary condition is given by

\[
B[\psi(r)]_{z=z_n} = 0, \quad n = 1, \ldots, N
\]

- Taking the Fourier transform to obtain the depth separated equation and inserting the value of the displacement potential,

\[
\frac{d^2}{dz^2} + (k^2 - k_x^2) G_\omega(k_x, z, z_s) = -\frac{\delta(z - z_s)}{2\pi}
\]

- \(G_\omega(k_x, z, z_s)\) is the depth dependent Green function, which is the superposition of the free field and homogeneous field.

- The total field must satisfy the boundary conditions.
Choosing the Cartesian coordinates and assuming an infinite line source along y-axis, the Helmholtz equation is reduced to two dimension, the range and the depth.

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(z) \right] \psi(x, z) = S_\omega \delta(x) \delta(z - z_s)
\]

The boundary condition is given by

\[
B[\psi(r)]_{z=z_n} = 0, \quad n = 1, \ldots, N
\]

Taking the Fourier transform to obtain the depth separated equation and inserting the value of the displacement potential,

\[
\left[ \frac{d^2}{dz^2} + (k^2 - k_x^2) \right] G_\omega(k_x, z, z_s) = -\frac{\delta(z - z_s)}{2\pi}
\]

\(G_\omega(k_x, z, z_s)\) is the depth dependent Green function, which is the superposition of the free field and homogeneous field.

The total field must satisfy the boundary conditions.
Plane propagation problem

Choosing the Cartesian coordinates and assuming an infinite line source along y-axis, the Helmholtz equation is reduced to two dimension, the range and the depth.

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(z) \right] \psi(x, z) = S_\omega \delta(x) \delta(z - z_s)
\]

The boundary condition is given by

\[ B[\psi(r)]_{z=z_n} = 0, \quad n = 1, \ldots, N \]

Taking the Fourier transform to obtain the depth separated equation and inserting the value of the displacement potential,

\[
\left[ \frac{d^2}{dz^2} + (k^2 - k_x^2) \right] G_\omega(k_x, z, z_s) = -\frac{\delta(z - z_s)}{2\pi}
\]

\( G_\omega(k_x, z, z_s) \) is the depth dependent Green function, which is the superposition of the free field and homogeneous field.

The total field must satisfy the boundary conditions.
Plane propagation problem

- Choosing the Cartesian coordinates and assuming an infinite line source along y-axis, the Helmholtz equation is reduced to two dimension, the range and the depth.

\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2(z) \right] \psi(x, z) = S_\omega \delta(x) \delta(z - z_s)
\]

- The boundary condition is given by

\[
B[\psi(r)]_{z=z_n} = 0, \quad n = 1, \ldots, N
\]

- Taking the Fourier transform to obtain the depth separated equation and inserting the value of the displacement potential,

\[
\left[ \frac{d^2}{dz^2} + (k^2 - k_x^2) \right] \omega(k_x, z, z_s) = -\frac{\delta(z - z_s)}{2\pi}
\]

- \(G_\omega(k_x, z, z_s)\) is the depth dependent Green function, which is the superposition of the free field and homogeneous field.

- The total field must satisfy the boundary conditions.
Reflection and Transmission

Assuming the simplest of the bottom model as shown in the figure. The point source is located in the water column at depth \( z = z_s \).

For homogeneous medium, the solution of the equation is given as

\[
H_\omega(k_r, z) = A^+(k_r)e^{ik_z z} + A^-(k_r)e^{-ik_z z}
\]

Where \( k_z \) is the vertical wavenumber, given by

\[
k_z = \sqrt{k^2 - k_r^2}
\]
Assuming the simplest of the bottom model as shown in the figure. The point source is located in the water column at depth $z = z_s$.

For homogeneous medium, the solution of the equation is given as

$$H_\omega(k_r, z) = A^+(k_r)e^{ik_z z} + A^-(k_r)e^{-ik_z z}$$

Where $k_z$ is the vertical wavenumber, given by

$$k_z = \sqrt{k^2 - k_r^2}$$
Assuming the simplest of the bottom model as shown in the figure. The point source is located in the water column at depth $z = z_s$.

For homogeneous medium, the solution of the equation is given as

$$H_\omega (k_r, z) = A^+(k_r)e^{ik_zz} + A^-(k_r)e^{-ik_zz}$$

Where $k_z$ is the vertical wavenumber, given by

$$k_z = \sqrt{k^2 - k_r^2}$$
Assuming the simplest of the bottom model as shown in the figure. The point source is located in the water column at depth $z = z_s$.

For homogeneous medium, the solution of the equation is given as

$$H_\omega(k_r, z) = A^+(k_r)e^{ik_zz} + A^-(k_r)e^{-ik_zz}$$

Where $k_z$ is the vertical wavenumber, given by

$$k_z = \sqrt{k^2 - k_r^2}$$
Using the radiation condition, the homogeneous solution in the upper halfspace is given by

\[ H_{\omega,1}(k_r, z) = A_1^-(k_r)e^{-ik_{z,1}z} \]

Similarly the solution in the lower halfspace will be

\[ H_{\omega,2}(k_r, z) = A_2^+(k_r)e^{ik_{z,2}z} \]

Both unknown amplitudes can be found using the boundary conditions of continuity of vertical displacement and continuity of pressure. The expression for the amplitudes is given by

\[ A_1^- = \frac{\rho_2 k_{z,1} - \rho_1 k_{z,2}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]

\[ A_2^+ = \frac{2\rho_1 k_{z,1}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]
Reflection and Transmission

- Using the radiation condition, the homogeneous solution in the upper halfspace is given by

\[ H_{\omega,1}(k_r, z) = A_1^-(k_r)e^{-ik_{z,1}z} \]

- Similarly the solution in the lower halfspace will be

\[ H_{\omega,2}(k_r, z) = A_2^+(k_r)e^{ik_{z,2}z} \]

- Both unknown amplitudes can be found using the boundary conditions of continuity of vertical displacement and continuity of pressure. The expression for the amplitudes is given by

\[ A_1^- = \frac{\rho_2k_{z,1} - \rho_1k_{z,2}}{\rho_2k_{z,1} + \rho_1k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]
\[ A_2^+ = \frac{2\rho_1k_{z,1}}{\rho_2k_{z,1} + \rho_1k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]
Using the radiation condition, the homogeneous solution in the upper halfspace is given by

\[ H_{\omega,1}(k_r, z) = A_1^-(k_r) e^{-ik_{z,1}z} \]

Similarly, the solution in the lower halfspace will be

\[ H_{\omega,2}(k_r, z) = A_2^+(k_r) e^{ik_{z,2}z} \]

Both unknown amplitudes can be found using the boundary conditions of continuity of vertical displacement and continuity of pressure. The expression for the amplitudes is given by

\[ A_1^- = \frac{\rho_2 k_{z,1} - \rho_1 k_{z,2}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]

\[ A_2^+ = \frac{2 \rho_1 k_{z,1}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]
Reflection and Transmission

- Using the radiation condition, the homogeneous solution in the upper halfspace is given by
  \[ H_{\omega,1}(k_r, z) = A_1^-(k_r)e^{-ik_{z,1}z} \]

- Similarly the solution in the lower halfspace will be
  \[ H_{\omega,2}(k_r, z) = A_2^+(k_r)e^{ik_{z,2}z} \]

- Both unknown amplitudes can be found using the boundary conditions of continuity of vertical displacement and continuity of pressure. The expression for the amplitudes is given by
  \[ A_1^- = \frac{\rho_2 k_{z,1} - \rho_1 k_{z,2}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]
  \[ A_2^+ = \frac{2\rho_1 k_{z,1}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]
Reflection and Transmission

- Using the radiation condition, the homogeneous solution in the upper halfspace is given by

\[ H_{\omega,1}(k_r, z) = A_1^-(k_r)e^{-i k_{z,1}z} \]

- Similarly the solution in the lower halfspace will be

\[ H_{\omega,2}(k_r, z) = A_2^+(k_r)e^{i k_{z,2}z} \]

- Both unknown amplitudes can be found using the boundary conditions of continuity of vertical displacement and continuity of pressure. The expression for the amplitudes is given by

\[ A_1^- = \frac{\rho_2 k_{z,1} - \rho_1 k_{z,2}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]

\[ A_2^+ = \frac{2\rho_1 k_{z,1}}{\rho_2 k_{z,1} + \rho_1 k_{z,2}} g_{\omega,1}(k_r, 0, z_s) \]
Hard Bottom Vs Soft Bottom

Figure: Spectral Domain for the hard bottom (top), soft bottom (bottom)
Figure: Reflection coefficient as a function of grazing angle for hard bottom (top), soft bottom (bottom). Solid curve: Magnitude. Dashed curve: Phase.
**Ideal fluid waveguide**

Assuming the simplest ocean waveguide with range independent, isovelocity water column and perfectly rigid boundaries as shown in the figure.

Solution of the waveguide problem can be obtained by superposition principle.
Assuming the simplest ocean waveguide with range independent, isovelocity water column and perfectly rigid boundaries as shown in the figure.

Solution of the waveguide problem can be obtained by superposition principle.
Assuming the simplest ocean waveguide with range independent, isovelocity water column and perfectly rigid boundaries as shown in the figure.

Solution of the waveguide problem can be obtained by superposition principle.
The field produced at point \((0, z_s)\) in the absence of boundaries is given by

\[
\psi(r, z) = -S_\omega \frac{e^{ikR}}{4\pi R}
\]

For solving the homogeneous equation which satisfies the boundary conditions, we can use two methods
- Image Method.
- Integral Transform solution.
The field produced at point \((0, z_s)\) in the absence of boundaries is given by

\[
\psi(r, z) = -S\omega \frac{e^{i k R}}{4\pi R}
\]

For solving the homogeneous equation which satisfies the boundary conditions, we can use two methods:
- Image Method.
- Integral Transform solution.
The field produced at point \((0, z_s)\) in the absence of boundaries is given by

\[
\psi(r, z) = -S\omega \frac{e^{ikR}}{4\pi R}
\]

For solving the homogeneous equation which satisfies the boundary conditions, we can use two methods

- Image Method.
- Integral Transform solution.
In the waveguide problem, sound will be multiply reflected between the two boundaries and require an infinite number of image sources.

Figure above shows the contribution from the physical source at depth $z_s$ and the first three image sources, leading to the first four terms in the expression in the total field

$$\psi(r, z) \approx -S\omega \frac{e^{ikR_{01}}}{R_{01}} \frac{e^{ikR_{02}}}{R_{02}} \frac{e^{ikR_{03}}}{R_{03}} + \frac{e^{ikR_{04}}}{R_{04}}$$
In the waveguide problem, sound will be multiply reflected between the two boundaries and require an infinite number of image sources.

Figure above shows the contribution from the physical source at depth $z_s$ and the first three image sources, leading to the first four terms in the expression in the total field

$$\psi(r, z) \approx -\frac{S_\omega}{4\pi} \left[ \frac{e^{ikR_{01}}}{R_{01}} - \frac{e^{ikR_{02}}}{R_{02}} - \frac{e^{ikR_{03}}}{R_{03}} + \frac{e^{ikR_{04}}}{R_{04}} \right]$$
In the waveguide problem, sound will be multiply reflected between the two boundaries and require an infinite number of image sources.

Figure above shows the contribution from the physical source at depth $z_s$ and the first three image sources, leading to the first four terms in the expression in the total field

$$\psi(r, z) \simeq -\frac{S \omega}{4\pi} \left[ \frac{e^{ikR_{01}}}{R_{01}} - \frac{e^{ikR_{02}}}{R_{02}} - \frac{e^{ikR_{03}}}{R_{03}} + \frac{e^{ikR_{04}}}{R_{04}} \right]$$
where the negative sign corresponds to an odd number of reflections and the positive signs correspond to an even number of reflections.

Expanding it to the total field, we have

$$\psi(r, z) = -\frac{S_\omega}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{e^{ikR_{m1}}}{R_{m1}} - \frac{e^{ikR_{m2}}}{R_{m2}} - \frac{e^{ikR_{m3}}}{R_{m3}} + \frac{e^{ikR_{m4}}}{R_{m4}} \right]$$
where the negative sign corresponds to an odd number of reflections and the positive signs correspond to an even number of reflections.

Expanding it to the total field, we have

$$
\psi(r, z) = -\frac{S_\omega}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{e^{ikR_{m1}}}{R_{m1}} - \frac{e^{ikR_{m2}}}{R_{m2}} - \frac{e^{ikR_{m3}}}{R_{m3}} + \frac{e^{ikR_{m4}}}{R_{m4}} \right]
$$
where the negative sign corresponds to an odd number of reflections and the positive signs correspond to an even number of reflections.

Expanding it to the total field, we have

$$\psi(r, z) = -S_\omega \frac{1}{4\pi} \sum_{m=0}^{\infty} \left[ \frac{e^{ikR_m}}{R_m} - \frac{e^{ikR_{m+1}}}{R_{m+1}} - \frac{e^{ikR_{m+2}}}{R_{m+2}} + \frac{e^{ikR_{m+3}}}{R_{m+3}} \right]$$
Computing the time domain green function, we have

$$g_t(r, z) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \left( \frac{\delta(R_{m1}/c - t)}{R_{m1}} - \frac{\delta(R_{m2}/c - t)}{R_{m2}} - \frac{\delta(R_{m3}/c - t)}{R_{m3}} + \frac{\delta(R_{m4}/c - t)}{R_{m4}} \right)$$

In order to obtain the received signal, the source function is convolved with the time domain green function.

At low frequencies, the multiples will interfere and the received signal will be distorted.

Only short and high frequency pulses can be individually identified as true images of the source signal.
Computing the time domain green function, we have

\[ g_t(r, z) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \left( \frac{\delta(R_{m1}/c - t)}{R_{m1}} - \frac{\delta(R_{m2}/c - t)}{R_{m2}} \right) \]

\[ - \frac{\delta(R_{m3}/c - t)}{R_{m3}} + \frac{\delta(R_{m4}/c - t)}{R_{m4}} \]
Computing the time domain green function, we have

\[ g_t(r, z) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \left( \frac{\delta(R_{m1}/c - t)}{R_{m1}} - \frac{\delta(R_{m2}/c - t)}{R_{m2}} \right) \]

\[ - \frac{\delta(R_{m3}/c - t)}{R_{m3}} + \frac{\delta(R_{m4}/c - t)}{R_{m4}} \]

In order to obtain the received signal, the source function is convolved with the time domain green function.

At low frequencies, the multiples will interfere and the received signal will be distorted.

Only short and high frequency pulses can be individually identified as true images of the source signal.
Computing the time domain green function, we have

\[ g_t(r, z) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \left( \frac{\delta(R_{m1}/c - t)}{R_{m1}} - \frac{\delta(R_{m2}/c - t)}{R_{m2}} \right) \]

\[ -\frac{\delta(R_{m3}/c - t)}{R_{m3}} + \frac{\delta(R_{m4}/c - t)}{R_{m4}} \]

In order to obtain the received signal, the source function is convolved with the time domain green function.

At low frequencies, the multiples will interfere and the received signal will be distorted.

Only short and high frequency pulses can be individually identified as true images of the source signal.
Computing the time domain green function, we have

\[ g_t(r, z) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \left( \frac{\delta(R_{m1}/c - t)}{R_{m1}} - \frac{\delta(R_{m2}/c - t)}{R_{m2}} ight) 
- \frac{\delta(R_{m3}/c - t)}{R_{m3}} + \frac{\delta(R_{m4}/c - t)}{R_{m4}} \]

In order to obtain the received signal, the source function is convolved with the time domain green function.

At low frequencies, the multiples will interfere and the received signal will be distorted.

Only short and high frequency pulses can be individually identified as true images of the source signal.
Using the integral transform technique, the total field is represented as

$$\psi(r, z) = \int_0^\infty \psi(k_r, z) J_0(k_r r) k_r dk_r$$

Using the superposition principle and applying the boundary conditions, the free-field waveguide solution becomes

$$\psi(k_r, z) = -\frac{S_\omega}{4\pi} \left\{ \begin{array}{ll}
\frac{\sin k_z z \sin k_z (D - z_s)}{k_z \sin k_z D}, & z < z_s \\
\frac{\sin k_z z_s \sin k_z (D - z)}{k_z \sin k_z D}, & z > z_s
\end{array} \right.$$

Using the relation between Hankel and Bessel function and doing some algebraic manipulations we have

$$\psi(r, z) = \frac{i S_\omega}{2D} \sum_{m=1}^\infty \sin(k_m z) \sin(k_m z_s) H_0^{(1)}(k_m r)$$
Integral Transform solution

Using the integral transform technique, the total field is represented as

\[ \psi(r, z) = \int_0^\infty \psi(k_r, z)J_0(k_r r)k_r dk_r \]

Using the superposition principle and applying the boundary conditions, the free-field waveguide solution becomes

\[ \psi(k_r, z) = -\frac{S_\omega}{4\pi} \begin{cases} \frac{\sin k_z z \sin k_z (D - z_s)}{k_z \sin k_z D}, & z < z_s \\ \frac{\sin k_z z_s \sin k_z (D - z)}{k_z \sin k_z D}, & z > z_s \end{cases} \]

Using the relation between Hankel and Bessel function and doing some algebraic manipulations we have

\[ \psi(r, z) = \frac{iS_\omega}{2D} \sum_{m=1}^{\infty} \sin(k_m z) \sin(k_m z_s) H_0^{(1)}(k_r m r) \]
Integral Transform solution

- Using the integral transform technique, the total field is represented as

\[ \psi(r, z) = \int_{0}^{\infty} \psi(k_r, z) J_0(k_r r) k_r dk_r \]

- Using the superposition principle and applying the boundary conditions, the free-field waveguide solution becomes

\[ \psi(k_r, z) = -\frac{S_\omega}{4\pi} \left\{ \frac{\sin k_z z \sin k_z (D - z_s)}{k_z \sin k_z D}, \quad z < z_s \right. \]
\[ \left. \quad \frac{\sin k_z z_s \sin k_z (D - z)}{k_z \sin k_z D}, \quad z > z_s \right\} \]

- Using the relation between Hankel and Bessel function and doing some algebraic manipulations we have

\[ \psi(r, z) = \frac{iS_\omega}{2D} \sum_{m=1}^{\infty} \sin(k_{zm} z) \sin(k_{zm} z_s) H_0^{(1)}(k_{rm} r) \]
Using the integral transform technique, the total field is represented as

\[
\psi(r, z) = \int_0^\infty \psi(k_r, z)J_0(k_r r)k_r dk_r
\]

Using the superposition principle and applying the boundary conditions, the free-field waveguide solution becomes

\[
\psi(k_r, z) = -\frac{S\omega}{4\pi} \begin{cases} 
\frac{\sin k_z z \sin k_z D - z_s}{k_z \sin k_z D}, & z < z_s \\
\frac{\sin k_z z_s \sin k_z (D - z)}{k_z \sin k_z D}, & z > z_s 
\end{cases}
\]

Using the relation between Hankel and Bassel function and doing some algebraic manipulations we have

\[
\psi(r, z) = \frac{iS\omega}{2D} \sum_{m=1}^{\infty} \sin(k_m z) \sin(k_m z_s) H_{1}^{(1)}(k_m r)
\]
Normal Mode solution

\[ \psi(r, z) = \frac{iS\omega}{2D} \sum_{m=1}^{\infty} \sin(k_m z_s)\sin(k_m z_s)H_0^{(1)}(k_m r) \]

- This is the normal mode expansion of the field.
- The field remains the same even if the source and the receiver are interchanged.
- Modal excitation is proportional to the amplitude of that particular mode at the source depth.
- Propagating Vs Evanescent modes.
Normal Mode solution

\[ \psi(r, z) = \frac{iS\omega}{2D} \sum_{m=1}^{\infty} \sin(k_z m z_s) \sin(k_z m z) H_0^{(1)}(k_r m r) \]

- This is the normal mode expansion of the field.
- The field remains the same even if the source and the receiver are interchanged.
- Modal excitation is proportional to the amplitude of that particular mode at the source depth.
- Propagating Vs Evanescent modes.
Normal Mode solution

\[ \psi(r, z) = \frac{iS\omega}{2D} \sum_{m=1}^{\infty} \sin(k_m z_0) \sin(k_m z_s) H_0^{(1)}(k_m r) \]

- This is the normal mode expansion of the field.
- The field remains the same even if the source and the receiver are interchanged.
- Modal excitation is proportional to the amplitude of that particular mode at the source depth.
- Propagating Vs Evanescent modes.
Normal Mode solution

\[ \psi(r, z) = \frac{i S \omega}{2D} \sum_{m=1}^{\infty} \sin(k_m z) \sin(k_m z_s) H_0^{(1)}(k_mr) \]

- This is the normal mode expansion of the field.
- The field remains the same even if the source and the receiver are interchanged.
- Modal excitation is proportional to the amplitude of that particular mode at the source depth.
- Propagating Vs Evanescent modes.
Normal Mode solution

\[ \psi(r, z) = \frac{i S \omega}{2 D} \sum_{m=1}^{\infty} \sin(k_m z) \sin(k_m z_s) H_0^{(1)}(k_m r) \]

- This is the normal mode expansion of the field.
- The field remains the same even if the source and the receiver are interchanged.
- Modal excitation is proportional to the amplitude of that particular mode at the source depth.
- Propagating Vs Evanescent modes.
A normal mode is the superposition of up and down going plane waves.

\[ \sin(k_{zm}z) = \frac{e^{ik_{zm}z} - e^{-ik_{zm}z}}{2i} \]

Both of the waves are propagating at the grazing angles \( \theta_m = \arctan(k_{zm}/k_{rm}) \). The propagation path is shown in the figure.

The dashed line shows the common wavefront for the wave passing through points A and B. The distance between A and B is given by \( L_{AB} = m\lambda \) where \( \lambda \) is the acoustic wavelength.

The discrete wavenumbers are the points where multiple reflections of the plane wave are in phase, giving rise to resonance.
A normal mode is the superposition of up and down going plane waves.

\[
\sin(k_{zm}z) = \frac{e^{ik_{zm}z} - e^{-ik_{zm}z}}{2i}
\]

Both of the waves are propagating at the grazing angles 
\(\theta_m = \arctan(k_{zm}/k_{rm})\). The propagation path is shown in the figure.

The dashed line shows the common wavefront for the wave passing through points A and B. The distance between A and B is given by 
\(L_{AB} = m\lambda\) where \(\lambda\) is the acoustic wavelength.

The discrete wavenumbers are the points where multiple reflections of the plane wave are in phase, giving rise to resonance.
A normal mode is the superposition of up and down going plane waves.

\[ \sin(k_{zm}z) = \frac{e^{ik_{zm}z} - e^{-ik_{zm}z}}{2i} \]

Both of the waves are propagating at the grazing angles \( \theta_m = \arctan(k_{zm}/k_{rm}) \). The propagation path is shown in the figure.

- The dashed line shows the common wavefront for the wave passing through points A and B. The distance between A and B is given by \( L_{AB} = m\lambda \) where \( \lambda \) is the acoustic wavelength.
- The discrete wavenumbers are the points where multiple reflections of the plane wave are in phase, giving rise to resonance.
A normal mode is the superposition of up and down going plane waves.

\[ \sin(k_{zm}z) = \frac{e^{ik_{zm}z} - e^{-ik_{zm}z}}{2i} \]

Both of the waves are propagating at the grazing angles \( \theta_m = \arctan(k_{zm}/k_{rm}) \). The propagation path is shown in the figure.

The dashed line shows the common wavefront for the wave passing through points A and B. The distance between A and B is given by \( L_{AB} = m\lambda \) where \( \lambda \) is the acoustic wavelength.

The discrete wavenumbers are the points where multiple reflections of the plane wave are in phase, giving rise to resonance.
A normal mode is the superposition of up and down going plane waves.

\[
\sin(k_{zm}z) = \frac{e^{ik_{zm}z} - e^{-ik_{zm}z}}{2i}
\]

Both of the waves are propagating at the grazing angles \( \theta_m = \arctan(k_{zm}/k_{rm}) \). The propagation path is shown in the figure.

The dashed line shows the common wavefront for the wave passing through points A and B. The distance between A and B is given by \( L_{AB} = m\lambda \) where \( \lambda \) is the acoustic wavelength.

The discrete wavenumbers are the points where multiple reflections of the plane wave are in phase, giving rise to resonance.
Deep Ocean waveguide

- Depth variations and temporal variations of the sound speed profile.
- The scale of temporal variations is much larger than the vertical variations.
- Range independent environment can provide a realistic model of the deep ocean specially for the arctic environment.
- In the exact solution, the water column is divided in multiple layers and the wave equation is solved for each layer.
- WKB solution approximates the solution of the wave equation by amplitude and phase where both are functions of depth.
Deep Ocean waveguide

- Depth variations and temporal variations of the sound speed profile.
- The scale of temporal variations is much larger than the vertical variations.
- Range independent environment can provide a realistic model of the deep ocean specially for the arctic environment.
- In the exact solution, the water column is divided in multiple layers and the wave equation is solved for each layer.
- WKB solution approximates the solution of the wave equation by amplitude and phase where both are functions of depth.
Deep Ocean waveguide

- Depth variations and temporal variations of the sound speed profile.
- The scale of temporal variations is much larger than the vertical variations.
- Range independent environment can provide a realistic model of the deep ocean specially for the arctic environment.
- In the exact solution, the water column is divided in multiple layers and the wave equation is solved for each layer.
- WKB solution approximates the solution of the wave equation by amplitude and phase where both are functions of depth.
Deep Ocean waveguide

- Depth variations and temporal variations of the sound speed profile.
- The scale of temporal variations is much larger than the vertical variations.
- Range independent environment can provide a realistic model of the deep ocean specially for the arctic environment.
- In the exact solution, the water column is divided in multiple layers and the wave equation is solved for each layer.
- WKB solution approximates the solution of the wave equation by amplitude and phase where both are functions of depth.
Depth variations and temporal variations of the sound speed profile.

The scale of temporal variations is much larger than the vertical variations.

Range independent environment can provide a realistic model of the deep ocean specially for the arctic environment.

In the exact solution, the water column is divided in multiple layers and the wave equation is solved for each layer.

WKB solution approximates the solution of the wave equation by amplitude and phase where both are functions of depth.
Depth variations and temporal variations of the sound speed profile. The scale of temporal variations is much larger than the vertical variations.

Range independent environment can provide a realistic model of the deep ocean specially for the arctic environment.

In the exact solution, the water column is divided in multiple layers and the wave equation is solved for each layer.

WKB solution approximates the solution of the wave equation by amplitude and phase where both are functions of depth.